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Inverse problem for a free transport equation using Carleman estimates

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This article is devoted to prove a stability result for an absorption coefficient for a free transport equation in a smooth domain Ω. The result is obtained using a global Carleman estimate with only one observation on a part of the boundary.

Keywords: Transport equation, Carleman estimates, stability.

AMS Subject Classification: 35L02, 35R30

1. Introduction

In this paper, we deal with the question of the identification of an absorption coefficient for a free transport equation in a bounded domain using Carleman estimates.
Some systems arising in Mathematical Biology, such as taxis-diffusion-reaction model, involve the radiative transport equation without scattering term (see [15] and the references therein). This equation describes the evolution of the density of the organism (cells, predators, parasitoids) in general. This paper is the first step in the study of inverse problems linked to angiogenesis process and this is why we do not consider the transport equation with an integral term.

Let Ω be a bounded open connected set in $\mathbb{R}^N$ whose boundary $\partial\Omega = \Gamma$ is assumed to be of class $\mathcal{C}^2$. We denote $\Omega_T := \Omega \times (0, T)$. Let $u(x, t) \in \mathbb{R}$ be the density of particles flow at time $t > 0$ and position $x \in \mathbb{R}^N$ with velocity $A = A(x)$. Let $\nu(x)$ be the outward normal unit vector to $\partial\Omega$ at $x \in \partial\Omega$. We define $\Gamma^{\pm}$ as

$$\Gamma^+ = \{x \in \partial\Omega\,;\ \nu(x) \cdot A > 0\} \text{ and } \Gamma^- = \{x \in \partial\Omega\,;\ \nu(x) \cdot A \leq 0\}.$$

Note that Γ^- represents the inflow part and Γ^+ the outflow part.

We consider the following problem

$$\begin{cases} \partial_t u(x,t) + A(x) \cdot \nabla u(x,t) + p(x)u(x,t) = 0, & \text{in } \ \Omega \times (0,T), \\ u(x,t) = h(x,t), \quad \text{on } \ \Gamma^- \times (0,T), \\ u(x,0) = u_0(x), \quad \text{in } \ \Omega. \end{cases} \tag{1.1}$$

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In the above problem, we suppose p, which is an absorption coefficient, to be in $L^\infty(\Omega)$, $A \in (W^{1,\infty}(\Omega))^N$ and (h, u_0) corresponds to the the boundary and the initial data lying in $L^2(\Gamma^- \times (0,T)) \times L^2(\Omega)$.
We also define the space $\mathcal{W}$ as follows:

$$\mathcal{W} = \Big\{u \in L^2(\Omega \times (0,T));\ \frac{\partial u}{\partial t} + A \cdot \nabla u \in L^2(\Omega \times (0,T))\Big\}.$$

It is well-known that, under the previous assumptions, (1.1) admits an unique solution which belongs to the space $\mathcal{W}$ and we have $u \in C([0,T]; L^2(\Omega))$ (see for example [12]).
Moreover, if $u_0 \in C^1(\Omega)$, $h \in C^1([0,T]; L^2(\Gamma^-))$ and satisfies the following compatibility conditions

$$u_0 = h|_{t=0}, \quad \partial_t h|_{t=0} + A \cdot \nabla u_0 + V u_0 = 0 \quad \text{on } \Gamma^-,$$

then

$$u \in C^1([0,T]; L^2(\Omega)), \quad A \cdot \nabla u \in C([0,T]; L^2(\Omega)).$$

Note that, from the maximum principle, if $u_0 \geq 0$ then $u \geq 0$ (see [12]).

Throughout this paper, we will denote by C a generic positive constant.

Our problem can be stated as follows:
Is it possible, under the previous assumptions, to determine the absorption coefficient $p(x)$ from the measurement of $(\partial_t u)|_{\Gamma^+ \times (0,T)}$?

The method based on Carleman estimates (see [14], [30])uses stronger geometrical assumptions, and in particular the following one:

Geometric condition:

$$\exists\ x_0 \notin \overline{\Omega}, \text{ such that } \Gamma^+ \supset \{x \in \partial\Omega,\ (x - x_0) \cdot \nu(x) \geq 0\}, \tag{1.2}$$

Below, we define the weight function we shall consider for the Carleman estimates.

Weight functions: Assume that Γ^+ satisfies (1.2) for some $x_0 \notin \overline{\Omega}$. Let $\beta \in (0,1)$ and define for $(x,t) \in \Omega \times (0,T)$

$$\psi(x,t) = |x - x_0|^2 - \beta t^2 + M, \text{ and for } \lambda > 0,\ \varphi(x,t) = e^{\lambda \psi(x,t)} \tag{1.3}$$

where M is choosen such that $\psi > 0$ in $\Omega \times (0,T)$ and β satisfies

$$T > \frac{1}{\sqrt{\beta}} \sup_{x \in \Omega} |x - x_0|. \tag{1.4}$$

Our main result is the following inequality:

There exists a constant $C > 0$ such that

$$||(p - \widetilde{p})(x)||_{L^2(\Omega)} \leq C||(\partial_t u - \partial_t \widetilde{u})(x,t)||_{L^2(\Gamma^+ \times (0,T))}.$$

where u (resp. $\widetilde{u}$) is a solution of (1.1) associated to $(p,\, h,\, u_0)$ (resp. $(\widetilde{p},\, h,\, u_0)$).

More precisely, see Theorem 3.1, Section 3.

The method of Carleman estimates has been introduced in the field of inverse problems by Bukhgeim and Klibanov, (see [7, 8, 17, 18]). Carleman estimate techniques are presented by Klibanov and Timonov in [21] for standard coefficients inverse problems both linear and nonlinear partial differential equations. These methods give a local Lipschitz stability around a single known solution. We can also cite some recent reviews on inverse problems and Carleman estimates by Choulli [11], Klibanov [16] and Yamamoto [29].
The transport equation plays an important role in physics. It includes areas such neutron transport, medical imaging and optical tomography see for example [2], [13], [9], [24]. Mathematical studies on the direct problem of (1.1) have been developped several times, for example, see [12], [23].
To our knowledge, there are some results in the study of inverse problems for the transport equation with an integral term. In [10], Choulli and Stefanov determine an absorption coefficient from the knowledge of the Albedo operator which gives a relationship between the two quantities $u|_{\Gamma^+}$ and $u|_{\Gamma^-}$. Their approach is based on the study of the singularities of the kernel of that operator.
The uniqueness and existence for coefficient inverse problems for the non stationary transport equation have been obtained by Prilepko and Ivankov [26] in a special form of coefficient using the overdetermination at a point. Some results on the overdetermined inverse problem for the transport equation can be found in the works of Tamasan [28] and Stefanov [27].
Stability of determining some coefficients is proved by Bal and Jolivet by the angularly average Albedo operator in [3] and by the full Albedo operator in [4]. In these papers, the authors have to make infinitely many measurements, the input-output can be limited to the boundary and the initial value can be zero. Bal in [1] and Stefanov in [27] have given a review of recent results on the inverse problem of the linear transport equation. Differents reconstructions are considered; uniqueness and stability results are proved in stationnary and non-stationnary case.
The approach of Klibanov and Pamyatnykh [20] is different from [3], [4]; indeed they measure a single output on $\Gamma^+ \times (0, T)$ with given initial value and boundary data on $\Gamma^- \times (0, T)$.
The inverse problem of reconstructing an absorption coefficient for the transport equation from available boundary measurements using Carleman estimates has been studied by Klibanov and Pamyatnykh in [20] and Machida and Yamamoto in [25]. In comparison with [25], we consider in our work the case when the velocity depends on the spatial variable x. Also, we use two large parameters instead of one in the Carleman estimate and energy estimates are needed for the proof of the stability result.
A Lipschitz stability estimate for the transport equation was also established by Klibanov and Pamyatnykh in [19] where the authors give a pointwise Carleman estimates. In control theory, such Carleman estimates have been used in order to obtain exact controlability by Klibanov and Yamamoto in [22].

The article is organized as follows. In Section 2 we establish a global Carleman estimate adapted to our problem and we prove some energy estimates. Section 3 is dedicated to the stability result for the absorption coefficient p.

2. Carleman estimates

In this section we will prove two Carleman estimates, one for the forward problem and another one for the backward problem.
We assume that the weight function ψ satisfies:

$$|(\partial_t \psi + A \cdot \nabla \psi)(x,t)| \neq 0, \quad \forall (x,t) \in \overline{\Omega} \times [-T,T]. \tag{2.1}$$

Let us give an example in which condition (2.1) is satisfied :
Suppose that $0 \notin \overline{\Omega}$ and $A(x) = x$. Then (2.1) is satisfied if

$$T < \frac{1}{\beta} \min_{\overline{\Omega}} |x|^2.$$

On the other hand, condition (1.4) requires

$$T > \frac{1}{\sqrt{\beta}} \max_{\overline{\Omega}} |x|.$$

Therefore, we obtain the following requirement for the parameter β

$$\sqrt{\beta} < \min_{\overline{\Omega}} |x|^2 \cdot \left(\max_{\overline{\Omega}} |x| \right)^{-1}.$$

We first give a Carleman estimate for the forward problem.

Proposition 2.1: *We assume that $p \in L^\infty(\Omega)$. Let ψ be the weight function defined by (1.3), satisfying (2.1). There exists s_0, λ_0 and a positive constant $C = C(s_0, \lambda_0, \Omega, T, \Gamma)$ such that for all $s > s_0$, $\lambda > \lambda_0$*

$$\|P_1(e^{s\varphi} v)\|_{L^2(\Omega_T)} + s\lambda^2 \int_0^T \int_\Omega \varphi |v|^2 e^{2s\varphi} dx dt \tag{2.2}$$

$$\leq C \int_0^T \int_\Omega |Lv|^2 e^{2s\varphi} dx dt + Cs\lambda \int_0^T \int_{\Gamma^+} \varphi |v|^2 A \cdot \nu e^{2s\varphi} d\sigma dt, \tag{2.3}$$

for all v such that $v \in L^2(\Omega \times (0,T))$ satisfying $Lv := \partial_t v + A \cdot \nabla v \in L^2(\Omega \times (0,T))$, $v|_{\Gamma^-} = 0$ and $v(\cdot, 0) = v(\cdot, T) = 0$, where P_1 is defined by (2.4) and (2.5).

Proof of Proposition 2.1 We set, for $s > 0$, $w(x,t) = e^{s\varphi(x,t)} v(x,t)$ and we introduce the operator

$$Pw = e^{s\varphi} L(e^{-s\varphi} w).$$

Then we have

$$Pw = \partial_t w + A \cdot \nabla w - s\lambda\varphi(\partial_t \psi + A \cdot \nabla \psi) w := P_1 w + P_2 w \tag{2.4}$$

where

$$P_1 w = \partial_t w + A \cdot \nabla w \quad \text{and} \quad P_2 w = -s\lambda\varphi(\partial_t \psi + A \cdot \nabla \psi) w. \tag{2.5}$$

Then

$$||Pw||^2_{L^2(\Omega_T)} = ||P_1 w||^2_{L^2(\Omega_T)} + ||P_2 w||^2_{L^2(\Omega_T)} + 2(P_1 w, P_2 w)_{L^2(\Omega_T)}. \tag{2.6}$$

We first look for a lower bound of $2(P_1 w, P_2 w)_{L^2(\Omega_T)}$. We essentially use integration by parts and the fact that $w(\cdot, 0) = w(\cdot, T) = 0$ and $w|_{\Gamma^-} = 0$.

$$2(P_1 w, P_2 w)_{L^2(\Omega_T)} = 2\int_0^T \int_\Omega (\partial_t w + A\cdot\nabla w)(-s\lambda\varphi(\partial_t\psi + A\cdot\nabla\psi)w)dxdt$$

$$= -2s\lambda \int_0^T \int_\Omega w\partial_t w\varphi(\partial_t\psi + A\cdot\nabla\psi)dxdt - 2s\lambda\int_0^T\int_\Omega w\nabla w\cdot A\varphi(\partial_t\psi + A\cdot\nabla\psi)dxdt$$

$$= -s\lambda\int_0^T\int_\Omega \partial_t(w^2)\varphi(\partial_t\psi + A\cdot\nabla\psi)dxdt - s\lambda\int_0^T\int_\Omega \nabla(w^2)\cdot A\varphi(\partial_t\psi + A\cdot\nabla\psi)dxdt$$

$$= I_1 + I_2.$$

A direct calculation leads to

$$I_1 = s\lambda^2\int_0^T\int_\Omega w^2\varphi\partial_t\psi(\partial_t\psi + A\cdot\nabla\psi)dxdt + s\lambda\int_0^T\int_\Omega w^2\varphi\partial_t(\partial_t\psi + A\cdot\nabla\psi)dxdt.$$

$$I_2 = s\lambda^2\int_0^T\int_\Omega \varphi\nabla\psi\cdot A(\partial_t\psi + A\cdot\nabla\psi)w^2dxdt + s\lambda\int_0^T\int_\Omega\varphi\nabla(A(\partial_t\psi + A\cdot\nabla\psi))w^2dxdt$$

$$-s\lambda\int_0^T\int_{\Gamma^+} w^2 A\cdot\nu(\partial_t\psi + A\cdot\nabla\psi)\varphi d\sigma dt.$$

By gathering the higher order terms and the lower order terms according to the powers of s and λ together, we obtain

$$s\lambda^2\int_0^T\int_\Omega w^2\varphi(\partial_t\psi + A\cdot\nabla\psi)^2dxdt - s\lambda\int_0^T\int_{\Gamma^+} w^2A\cdot\nu(\partial_t\psi + A\cdot\nabla\psi)\varphi d\sigma dt$$

$$+ s\lambda\int_\Omega\int_0^T w^2\varphi\Big(\partial_t^2\psi + 2A\cdot\nabla\partial_t\psi + |A|^2\Delta\psi + \nabla\cdot A(\partial_t\psi + A\cdot\nabla\psi)\Big)dxdt = 2(P_1w, P_2w)_{L^2(\Omega_T)}.$$

We focus on the dominating term (higher powers in s and λ), we want it to be positive. Using the regularity assumptions on ψ and A and condition (2.1), we get

$$\int_0^T\int_\Omega |P_1 w|^2dxdt + s\lambda^2\int_0^T\int_\Omega\varphi w^2dxdt \le C\int_0^T\int_\Omega|Pw|^2dxdt + Cs\lambda\int_0^T\int_\Omega\varphi w^2dxdt$$
$$+ Cs\lambda\int_0^T\int_{\Gamma^+} w^2A\cdot\nu\varphi d\sigma dt.$$

For large $s > 0$ and $\lambda > 0$, the last integral of the right hand side is absorbed by the dominating term in $s\lambda^2$ of the left hand side, it follows

$$\int_0^T\int_\Omega|P_1w|^2dxdt + s\lambda^2\int_0^T\int_\Omega w^2\varphi dxdt \le C\int_0^T\int_\Omega|Pw|^2dxdt + Cs\lambda\int_0^T\int_{\Gamma^+}w^2A\cdot\nu\varphi d\sigma dt.$$

Going back to v, we conclude the proof.

■

Now, we give the Carleman estimate for the backward problem.

Proposition 2.2: *We assume that $p \in L^\infty(\Omega)$. Let ψ be the weight function defined by (1.3), satisfying (2.1). There exists s_0, λ_0 and a positive constant $C = C(s_0, \lambda_0, \Omega, T, \Gamma)$ such that for all $s > s_0$, $\lambda > \lambda_0$*

$$\begin{aligned} s\lambda^2 \int_0^T \int_\Omega \varphi |v|^2 e^{2s\varphi} dxdt \leq C \int_0^T \int_\Omega |L_{back} v|^2 e^{2s\varphi} dxdt \\ + Cs\lambda \int_0^T \int_{\Gamma^+} \varphi |v|^2 A \cdot \nu e^{2s\varphi} d\sigma dt. \end{aligned} \tag{2.7}$$

for all v such that $v \in L^2(\Omega \times (0,T))$ satisfying $L_{back} v := -\partial_t v + A \cdot \nabla v \in L^2(\Omega \times (0,T))$, $v|_{\Gamma^-} = 0$ and $v(\cdot, 0) = v(\cdot, T) = 0$.

Proof of Proposition 2.2
In the same way as for the forward problem, we estimate the scalar product $(P_{1,back} w, P_{2,back} w)_{L^2(\Omega_T)}$, we obtain

$$2(P_{1,back} w, P_{2,back} w)_{L^2(\Omega_T)} = s\lambda^2 \int_0^T \int_\Omega w^2 \varphi (\partial_t \psi - A \cdot \nabla \psi)^2 dxdt$$

$$+ s\lambda \int_0^T \int_\Omega w^2 \varphi (\partial_t^2 \psi - 2A \cdot \nabla \psi + |A|^2 \Delta \psi - \nabla \cdot A(\partial_t \psi - A \cdot \nabla \psi)) dxdt$$

$$+ s\lambda \int_0^T \int_{\Gamma^+} w^2 A \cdot \nu (\partial_t \psi - A \cdot \nabla \psi) \varphi \, d\sigma dt.$$

In this case, we do not know the sign of $\partial_t \psi - A \cdot \nabla \psi$, we can only say, from (2.1), that $\partial_t \psi - A \cdot \nabla \psi \neq 0$. So, we have

$$s\lambda^2 \int_0^T \int_\Omega w^2 \varphi dxdt \leq \int_0^T \int_\Omega |P_{back} w|^2 dxdt + Cs\lambda \int_0^T \int_{\Gamma^+} w^2 A \cdot \nu \varphi d\sigma dt.$$

Going back to v, we get (2.7).

■

Now with the two Carleman estimates (2.2) and (2.7), we give the following estimate we shall use for the stability result.

Proposition 2.3: *We assume that $p \in L^\infty(\Omega)$. Let ψ be the weight function defined by (1.3), satisfying (2.1). We set*

$$\widehat{A}(x,t) = \begin{cases} A(x) & \text{if} \quad t \in (0,T), \\ -A(x) & \text{if} \quad t \in (-T,0). \end{cases} \tag{2.8}$$

Then, there exists s_0, λ_0 and a positive constant $C = C(s_0, \lambda_0, \Omega, T, \Gamma)$ such that *for all* $s > s_0$*,* $\lambda > \lambda_0$

$$\|P_1(e^{s\varphi}v)\|_{L^2(\Omega\times(-T,T))} + s\lambda^2\int_{-T}^{T}\int_{\Omega}\varphi|v|^2e^{2s\varphi}dxdt \le C\int_{-T}^{T}\int_{\Omega}|Lv|^2e^{2s\varphi}dxdt$$
$$+Cs\lambda\int_{-T}^{T}\int_{\Gamma^+}\varphi|v|^2\widehat{A}\cdot\nu e^{2s\varphi}d\sigma dt, \tag{2.9}$$

for all v *such that* $v \in L^2(\Omega\times(-T,T))$ *satisfying* $Lv := \partial_t v + \widehat{A}\cdot\nabla v \in L^2(\Omega\times(-T,T))$*,* $v|_{\Gamma^-} = 0$ *and* $v(\cdot,-T) = v(\cdot,T) = 0$*.*

3. Stability result

In this section, we give a stability and a uniqueness result for the absorption coefficient $p(x)$. In the perspective of numerical reconstruction, such problems are ill-posed and thus, the stability results are important. For the proof of our main result, we use both local and global Carleman estimates and energy estimates. Such weighted energy estimates have been proven in [5] for the wave equation in a bounded domain.

Theorem 3.1 : *Let* u *(resp.* $\widetilde{u}$*) be a solution of* (1.1) *associated to* $(p,\, h,\, u_0)$ *(resp.* $(\widetilde{p},\, h,\, u_0)$*). Let* ψ *be the weight function defined by (1.3), satisfying (2.1). Then, there exists a constant* $C > 0$ *such that*

$$\int_{\Omega}|(p-\widetilde{p})(x)|^2dx \le C\int_0^T\int_{\Gamma^+}|(\partial_t u - \partial_t\widetilde{u})(x,t)|^2d\sigma dt$$

Proof of Theorem 3.1 For the proof of our main result, we will proceed in several steps.

Step 1. We linearize our problem. We set $U = u - \widetilde{u}$ and we extend $\partial_t U$ as follows :

$$Y(x,t) = \begin{cases}\partial_t U(x,t) & t > 0,\\ \partial_t U(x,-t) & t < 0.\end{cases}$$

Then Y is solution of

$$\begin{cases}\partial_t Y + \widehat{A}(x,t)\cdot\nabla Y + p(x)Y = (\widetilde{p}-p)(x)\partial_t\widetilde{u} & \text{in } \Omega\times(-T,T),\\ Y(x,t) = 0 & \text{on } \Gamma^-\times(-T,T),\\ Y(x,0) = (\widetilde{p}-p)(x)u_0(x) & \text{in } \Omega.\end{cases} \tag{3.1}$$

where $\widehat{A}$ is defined by (2.8).

Step 2. Since Y does not satisfy the required hypothesis $Y(\cdot,-T) = Y(\cdot,T) = 0$, in order to apply Proposition 2.3, we introduce a cut-off function $\chi \in C_c^\infty(\mathbb{R})$ such that $0 \le \chi \le 1$. We choose $\eta \in (0,T)$ such that $\psi(x,t) \le C \le \psi(x,0)$, $\forall t \in (-T,-T+\eta)\cup(T-\eta,T)$ and we define

$$\chi(t) = \begin{cases}1, & \text{if } -T+\eta \le t \le T-\eta,\\ 0, & \text{if } t \le -T \text{ or } t \ge T.\end{cases}$$

We set $\widetilde{Y} = \chi Y$. Therefore, we can apply the Carleman estimate (2.9) to $\widetilde{Y}$ which is the solution to the following problem

$$\begin{cases} \partial_t \widetilde{Y} + \widehat{A}(x,t) \cdot \nabla \widetilde{Y} + p(x)\widetilde{Y} = \chi LY + Y\partial_t \chi & \text{in } \Omega \times (-T,T), \\ \widetilde{Y}(x,t) = 0 & \text{on } \Gamma^- \times (-T,T), \\ \widetilde{Y}(x,-T) = \widetilde{Y}(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{3.2}$$

We obtain

$$\|P_1(e^{s\varphi}\widetilde{Y})\|_{L^2(\Omega\times(-T,T))} + s\lambda^2 \int_{-T}^{T}\int_\Omega \varphi|\widetilde{Y}|^2 e^{2s\varphi}dxdt \leq C\int_{-T}^{T}\int_\Omega |\chi LY + Y\partial_t\chi|^2 e^{2s\varphi}dxdt$$
$$+Cs\lambda \int_{-T}^{T}\int_{\Gamma^+} \varphi|\widetilde{Y}|^2 \widehat{A}\cdot \nu e^{2s\varphi}d\sigma dt.$$

Note that $supp\, \partial_t\chi \subset (-T,-T+\eta)\cup(T-\eta,T)$, then we get the following estimate for Y

$$\|P_1(e^{s\varphi}Y)\|_{L^2(\Omega\times(-T+\eta,T-\eta))} + s\lambda^2 \int_{\eta}^{T-\eta}\int_\Omega \varphi|Y|^2 e^{2s\varphi}dxdt \leq C\int_{0}^{T}\int_\Omega |LY|^2 e^{2s\varphi}dxdt$$
$$+Cs\lambda \int_{-T}^{T}\int_{\Gamma^+} \varphi|\widetilde{Y}|^2 \widehat{A}\cdot \nu e^{2s\varphi}d\sigma dt + C\int_{-T}^{-T+\eta}\int_\Omega |Y|^2 e^{2s\varphi}dxdt + C\int_{T-\eta}^{T}\int_\Omega |Y|^2 e^{2s\varphi}dxdt. \tag{3.3}$$

Step 3. Here, we want to give an estimation of the last two integrals of the right hand side of (3.3) by the integral of the left hand side of the same inequality. The aim is to absorb the last two terms in the right hand side of (3.3) by the left hand side, for s large enough. In order to do that, we establish some energy estimates. First, we fix $\lambda = \lambda_0$ and use the fact that φ is bounded from below by 1 and from above by some constants depending on λ. Then, we define the following weighted energy:

$$E(t) = \frac{1}{2}\int_\Omega |Y|^2 e^{2s\varphi}dx.$$

- **Estimation of** $\displaystyle\int_{T-\eta}^{T}\int_\Omega |Y|^2 e^{2s\varphi}dxdt$

We calculate

$$\frac{dE}{dt} = s\int_\Omega \partial_t\varphi|Y|^2 e^{2s\varphi}dx + \int_\Omega |Y|\partial_t Y e^{2s\varphi}dx$$
$$= s\int_\Omega \partial_t\varphi|Y|^2 e^{2s\varphi}dx + \int_\Omega (LY - \widehat{A}\cdot\nabla Y)Y e^{2s\varphi}dx.$$

Then, we have

$$\frac{dE}{dt} - s\int_\Omega \partial_t\varphi|Y|^2 e^{2s\varphi}dx + \frac{1}{2}\int_\Omega e^{2s\varphi}\widehat{A}\cdot\nabla(|Y|^2)dx = \int_\Omega YLYe^{2s\varphi}dx.$$

After integration by parts, we get

$$\begin{aligned} &\frac{dE}{dt} - s\int_\Omega (\partial_t\varphi + \nabla\varphi\cdot\widehat{A})|Y|^2 e^{2s\varphi}dx + \frac{1}{2}\int_{\Gamma^+}\widehat{A}\cdot\nu|Y|^2e^{2s\varphi}d\sigma \\ &= \int_\Omega YLYe^{2s\varphi}dx + \frac{1}{2}\int_\Omega \nabla\cdot\widehat{A}|Y|^2e^{2s\varphi}dx. \end{aligned} \tag{3.4}$$

Moreover for all large $s>0$, since $-(\partial_t\varphi + \nabla\varphi\cdot\widehat{A}) \geq c > 0$, we obtain

$$\frac{dE}{dt} + sc\int_\Omega |Y|^2e^{2s\varphi}dx \leq \int_\Omega YLYe^{2s\varphi}dx. \tag{3.5}$$

Using the formula $2ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}$ with $\varepsilon = sc$, we estimate the right hand side as follows:

$$\Big|\int_\Omega YLYe^{2s\varphi}dx\Big| \leq \frac{1}{2}sc\int_\Omega |Y|^2e^{2s\varphi}dx + \frac{1}{2sc}\int_\Omega |LY|^2e^{2s\varphi}dx.$$

Substituing this estimate in (3.5), we have

$$\frac{dE}{dt} + scE(t) \leq \frac{1}{2sc}\int_\Omega |LY|^2e^{2s\varphi}dx.$$

On the other hand, for $t\in(T-\eta,T)$, using the Gronwall lemma, we get

$$\begin{aligned} E(t) &\leq e^{\int_{T-\eta}^t -csd\tau}\Big(E(T-\eta) + \int_{T-\eta}^t \frac{1}{2sc}\int_\Omega e^{2s\varphi(\tau)}|LY(\tau)|^2dxd\tau\Big) \\ &\leq e^{-sc(t-(T-\eta))}E(T-\eta) + \frac{e^{sc(T-t-\eta)}}{2sc}\int_{T-\eta}^t\int_\Omega e^{2s\varphi(\tau)}|LY(\tau)|^2dxd\tau \\ &\leq e^{-sc(t-(T-\eta))}E(T-\eta) + \frac{1}{2sc}\int_{T-\eta}^T\int_\Omega e^{2s\varphi(\tau)}|LY(\tau)|^2dxd\tau. \end{aligned}$$

Integrating this relation for t between $T-\eta$ and T , we obtain:

$$\begin{aligned} \int_{T-\eta}^T E(t)dt &\leq E(T-\eta)\int_T^{T-\eta} e^{-sc(t-(T-\eta))}dt + \int_{T-\eta}^T\frac{1}{2sc}\int_{T-\eta}^T\int_\Omega e^{2s\varphi(\tau)}|LY(\tau)|^2dxd\tau dt \\ &\leq E(T-\eta)\int_T^{T-\eta} e^{-sc(t-(T-\eta))}dt + \frac{\eta}{2sc}\int_{-T}^T\int_\Omega e^{2s\varphi}|LY|^2dxdt, \end{aligned}$$

and thus

$$\int_{T-\eta}^T E(t)dt \leq \frac{C}{s}E(T-\eta) + \frac{C}{s}\int_{-T}^T\int_\Omega e^{2s\varphi}|LY|^2dxdt. \tag{3.6}$$

Now, we want to estimate $E(T-\eta)$ by $E(\tau)$ for $\tau\in(\eta,T-\eta)$. We use (3.4) and we integrate between τ and $T-\eta$, this leads to

$$\int_\tau^{T-\eta} \frac{dE}{dt}dt + \frac{1}{2}\int_\tau^{T-\eta}\int_{\Gamma^+} \widehat{A}\cdot\nu|Y|^2 e^{2s\varphi}d\sigma dt$$

$$= \int_\tau^{T-\eta} s\int_\Omega (\partial_t\varphi + \widehat{A}\cdot\nabla\varphi)|Y|^2 e^{2s\varphi}dxdt + \frac{1}{2}\int_\tau^{T-\eta}\int_\Omega \nabla\cdot\widehat{A}|Y|^2 e^{2s\varphi}dxdt$$

$$+ \int_\tau^{T-\eta}\int_\Omega YLYe^{2s\varphi}dxdt.$$

Then, using the Cauchy-Schwarz inequality, we have

$$\int_\tau^{T-\eta} \frac{dE}{dt}dt + \frac{1}{2}\int_\tau^{T-\eta}\int_{\Gamma^+} \widehat{A}\cdot\nu|Y|^2 e^{2s\varphi}d\sigma dt$$

$$\le Cs\int_\tau^{T-\eta}\int_\Omega |Y|^2 e^{2s\varphi}dxdt + \frac{1}{2}sc\int_\tau^{T-\eta}\int_\Omega |Y|^2 e^{2s\varphi}dxdt + \frac{1}{2sc}\int_\tau^{T-\eta}\int_\Omega |LY|^2 e^{2s\varphi}dxdt.$$

It follows that

$$E(T-\eta) - E(\tau) \le Cs\int_\eta^{T-\eta}\int_\Omega |Y|^2 e^{2s\varphi}dxdt + \frac{C}{s}\int_{-T}^{T}\int_\Omega |LY|^2 e^{2s\varphi}dxdt.$$

Integrating between η and $T-\eta$, we obtain, for $s>0$ sufficiently large,

$$E(T-\eta) \le Cs\int_\eta^{T-\eta} E(t)dt + \frac{C}{s}\int_{-T}^{T}\int_\Omega |LY|^2 e^{2s\varphi}dxdt. \tag{3.7}$$

Finally, thanks to (3.6) and (3.7), we obtain

$$\int_{T-\eta}^{T}\int_\Omega |Y|^2 e^{2s\varphi}dxdt \le C\int_\eta^{T-\eta}\int_\Omega |Y|^2 e^{2s\varphi}dxdt + \frac{C}{s}\int_{-T}^{T}\int_\Omega e^{2s\varphi}|LY|^2 dxdt. \tag{3.8}$$

- **Estimation of** $\displaystyle\int\limits_{-T}^{-T+\eta}\int\limits_{\Omega} |Y|^2 e^{2s\varphi}dxdt$

Let t be in $(-T, -T+\eta)$. We would like to obtain the same result as previously. Therefore, we make the change of variables $t \to -t$, we introduce $Y_{back}(x,t) = Y(x,-t)$ and apply the above estimates to Y_{back}. Thus, (3.6), (3.7) coincide with the following ones:

$$\int_{-T}^{-T+\eta} E(t)dt \le \frac{C}{s}E(-T+\eta) + \frac{C}{s}\int_{-T}^{T}\int_\Omega e^{2s\varphi}|LY|^2 dxdt. \tag{3.9}$$

$$E(-T+\eta) \le Cs\int_{-T+\eta}^{T-\eta} E(t)dt + \frac{C}{s}\int_{-T}^{T}\int_{\Omega} |LY|^2 e^{2s\varphi} dxdt. \tag{3.10}$$

Finally, thanks to (3.9) and (3.10), we obtain

$$\int_{-T}^{-T+\eta}\int_{\Omega} |Y|^2 e^{2s\varphi} dxdt \le C\int_{-T+\eta}^{T-\eta}\int_{\Omega} |Y|^2 e^{2s\varphi} dxdt + \frac{C}{s}\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|LY|^2 dxdt. \tag{3.11}$$

Now, using (3.8) and (3.11) in (3.3), we obtain:

$$s\int_{-T+\eta}^{T-\eta}\int_{\Omega} |Y|^2 e^{2s\varphi} dxdt \le C\int_{-T}^{T}\int_{\Omega} |LY|^2 e^{2s\varphi} dxdt + Cs\int_{-T}^{T}\int_{\Gamma^+} |Y|^2 \widehat{A}\cdot\nu e^{2s\varphi} d\sigma dt. \tag{3.12}$$

We deduce from (3.7), (3.8), (3.12), the following Carleman estimate for Y

$$\|P_1(e^{s\varphi}Y)\|_{L^2(\Omega\times(-T+\eta,T-\eta))} + s\int_{-T}^{T}\int_{\Omega} |Y|^2 e^{2s\varphi} dxdt \tag{3.13}$$

$$\le C\int_{-T}^{T}\int_{\Omega} |LY|^2 e^{2s\varphi} dxdt + Cs\int_{-T}^{T}\int_{\Gamma^+} |Y|^2 \widehat{A}\cdot\nu e^{2s\varphi} d\sigma dt.$$

Remark 1 : In (3.13), we obtain a lower bound of $||e^{s\varphi}P_1Y||_{L^2(\Omega\times(-T+\eta,T-\eta))}$ but we can obtain it on $(\Omega\times(-T,T))$. Since $P_1Y = \partial_t Y + \widehat{A}\cdot\nabla Y = -pY - (p-\widetilde{p})\partial_t\widetilde{u}$, we have

$$\int_{-T}^{-T+\eta}\int_{\Omega} |P_1Y|^2 e^{2s\varphi} dxdt \le C\int_{-T}^{-T+\eta}\int_{\Omega} |Y|^2 e^{2s\varphi} dxdt + C\int_{-T}^{-T+\eta}\int_{\Omega} |p-\widetilde{p}|^2|\partial_t\widetilde{u}|^2 e^{2s\varphi} dxdt,$$

and also

$$\int_{T-\eta}^{T}\int_{\Omega} |P_1Y|^2 e^{2s\varphi} dxdt \le C\int_{T-\eta}^{T}\int_{\Omega} |Y|^2 e^{2s\varphi} dxdt + C\int_{T-\eta}^{T}\int_{\Omega} |p-\widetilde{p}|^2|\partial_t\widetilde{u}|^2 e^{2s\varphi} dxdt.$$

Finally, (3.13) yields the following estimate

$$\|P_1(e^{s\varphi}Y)\|_{L^2(\Omega\times(-T,T))} + s\int_{-T}^{T}\int_{\Omega} |Y|^2 e^{2s\varphi} dxdt + \int_{-T}^{T}\int_{\Omega} |P_1Y|^2 e^{2s\varphi} dxdt \tag{3.14}$$

$$\le C\int_{-T}^{T}\int_{\Omega} |p-\widetilde{p}|^2|\partial_t\widetilde{u}|^2 e^{2s\varphi} dxdt + Cs\int_{-T}^{T}\int_{\Gamma^+} |Y|^2 \widehat{A}\cdot\nu e^{2s\varphi} d\sigma dt.$$

Step 4. Stability result

Let $W = e^{s\varphi}\widetilde{Y}$. Recall that $P_1 W = \partial_t W + \widehat{A}\cdot\nabla W$ and consider the following integral $\mathcal{I} = \displaystyle\int_{-T}^{0}\int_{\Omega} P_1 W \cdot W\,dx\,dt$.
Following the method introduced in [6], we give an upper bound of $\mathcal{I}$ using Carleman estimate.

$$\begin{aligned}|\mathcal{I}| &= |\int_{-T}^{0}\int_{\Omega} P_1 W\cdot W dxdt|\\ &\le s^{-\frac{1}{2}}\Big(\int_{-T}^{0}\int_{\Omega}|P_1W|^2dxdt\Big)^{\frac{1}{2}}\Big(s\int_{-T}^{0}\int_{\Omega}|Y|^2e^{2s\varphi}\Big)^{\frac{1}{2}}dxdt.\end{aligned}$$

Applying Young inequality, we have

$$|\mathcal{I}| \le s^{-\frac{1}{2}}\Big(\int_{-T}^{T}\int_{\Omega}|P_1W|^2dxdt + s\int_{-T}^{T}\int_{\Omega}|Y|^2e^{2s\varphi}dxdt\Big).$$

Using (3.14), we obtain

$$|\mathcal{I}| \le Cs^{-\frac{1}{2}}\left(\int_{-T}^{T}\int_{\Omega}|p-\widetilde{p}|^2|\partial_t\widetilde{u}|^2dxdt + s\int_{\Gamma^+}\int_{-T}^{T}|Y|^2\widehat{A}\cdot\nu e^{2s\varphi}d\sigma dt\right). \tag{3.15}$$

Now, let's compute $\mathcal{I}$. After integration by parts, we have

$$\frac{1}{2}\int_{\Omega}|Y(x,0)|^2e^{2s\varphi(x,0)}dx = \mathcal{I} + \frac{1}{2}\int_{-T}^{0}\int_{\Omega}(\nabla\cdot\widehat{A})|\widetilde{Y}|^2e^{2s\varphi}dxdt - \int_{-T}^{0}\int_{\Gamma^+}\widehat{A}\cdot\nu|\widetilde{Y}|^2e^{2s\varphi}d\sigma dt.$$

With (3.15) and (3.14), we obtain

$$\begin{aligned}\frac{1}{2}\int_{\Omega}|Y(x,0)|^2e^{2s\varphi(x,0)}dx &\le C(s^{-\frac{1}{2}}+s^{-1})\Big\{\int_{-T}^{T}\int_{\Omega}|p-\widetilde{p}|^2|\partial_t\widetilde{u}|^2e^{2s\varphi}dxdt\Big\}\\ &\quad + C(s^{\frac{1}{2}}+1)\int_{-T}^{T}\int_{\Gamma^+}\widehat{A}\cdot\nu|Y|^2e^{2s\varphi}d\sigma dt.\end{aligned}$$

On the other hand, we have

$$Y(x,0) = \partial_t U(x,0) = -(p-\widetilde{p})(x)\widetilde{u}(x,0).$$

Substituting Y in the last inequality, we get

$$\begin{aligned}&\frac{1}{2}\int_{\Omega}|(p-\widetilde{p})(x)\widetilde{u}(x,0)|^2e^{2s\varphi(x,0)}dx \le C(s^{-\frac{1}{2}}+s^{-1})\int_{-T}^{T}\int_{\Omega}|p-\widetilde{p}|^2|\partial_t\widetilde{u}|^2e^{2s\varphi}dxdt\\ &+C(s^{\frac{1}{2}}+1)\int_{-T}^{T}\int_{\Gamma^+}\widehat{A}\cdot\nu|Y|^2e^{2s\varphi}d\sigma dt.\end{aligned} \tag{3.16}$$

Remark 2 : Recall that $x_0 \notin \overline{\Omega}$, so by construction φ is strictly bounded from below. Moreover $e^{2s\varphi(x,t)} \le e^{2s\varphi(x,0)}$ for all $x\in\Omega$ and $t\in(-T,T)$. From $\widetilde{u}\in$

$W^{1,2}(-T,T;L^\infty(\Omega))$, we have

$$\exists k_0 \in L^2(-T,T), |\partial_t \widetilde{u}(x,t)| \leq k_0(t)|\widetilde{u}(x,0)|, \ \forall x \in \Omega, \ t \in (-T,T).$$

Using the previous remark, from (3.16), it follows

$$\Big(\frac{1}{2} - C(s^{-\frac{1}{2}} + s^{-1})\Big) \int_\Omega |(p-\widetilde{p})(x)|^2 |\widetilde{u}(x,0)|^2 e^{2s\varphi(x,0)} dx \leq C(s^{\frac{1}{2}}+1) \int_{-T}^T \int_{\Gamma^+} \widehat{A} \cdot \nu |Y|^2 e^{2s\varphi} d\sigma dt.$$

Then, if s is large enough, we deduce that there exists a constant $C = C(s_0, \lambda_0, \Omega, T, \Gamma) > 0$ such that

$$\int_\Omega |(p-\widetilde{p})(x)|^2 |\widetilde{u}(x,0)|^2 e^{2s\varphi(x,0)} dx \leq C \int_{-T}^T \int_{\Gamma^+} \widehat{A} \cdot \nu |Y|^2 e^{2s\varphi} d\sigma dt.$$

Under the conditions satisfied by ψ, $\widehat{A}$, we note that $\widehat{A} \cdot \nu$ and $e^{2s\varphi}$ are bounded on $(-T,T) \times \Gamma^+$. Indeed, we denote

$$a = \min_{\overline{\Omega}} [exp(s\varphi(x,0))],$$

$$b = \max_{\overline{\Omega}\times[0,T]} [exp(s\varphi(x,t))].$$

Therefore, since $|\widetilde{u}(x,0)| \geq r_0 > 0$ in $\overline{\Omega}$, we obtain the following stability result

$$\int_\Omega |(p-\widetilde{p})(x)|^2 dx \leq C \int_{-T}^T \int_{\Gamma^+} |Y|^2 e^{2s\varphi} d\sigma dt.$$

where $C = C(\frac{b}{a}, s_0, \lambda_0, \Omega, T, \Gamma)$.

Since Y is an extension to $\partial_t U := \partial_t u - \partial_t \widetilde{u}$ for $t < 0$, we have

$$\int_\Omega |p - \widetilde{p}|^2 dx \leq C \int_0^T \int_{\Gamma^+} |\partial_t u - \partial_t \widetilde{u}|^2 d\sigma dt$$

and the proof of Theorem 3.1 is completed.

■

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